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SAINT-VENANT PROBLEM FOR SOLIDS WITH HELICAL RHOMBOHEDRAL ANISOTROPY. TENSION–TORSION PROBLEMS

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By methods of homogeneous solutions and the spectral theory of operators, the construction of solutions of the Saint-Venant problems of tension–torsion of a cylindrical tube with helical anisotropy is reduced to integration of boundary-value problems for ordinary differential equations with variable coefficients. The solutions are constructed by analytical and numerical methods. Elements of the stiffness matrix and the stress-strain state are analyzed, depending on the problem parameter.

Key words: *Saint-Venant problem, stiffness, helical anisotropy, sweep method.*

The present paper describes the construction of solutions of the Saint-Venant problem of tension and torsion of a cylinder possessing helical rhombohedral anisotropy. By methods of the spectral theory of operators [1–6], the problems are reduced to integration of boundary-value problems for ordinary differential equations. An analytical solution is obtained for a particular case of cylindrical rhombohedral anisotropy. The solutions of problems for a cylinder with helical rhombohedral anisotropy are constructed by two methods: for small values of the dimensionless parameter $\tau_0 = a\tau$, where a is the cylinder radius and τ is the "torsion" (characteristic of helical anisotropy), the analytical solution is constructed by the method of the small parameter; for arbitrary values of τ, the solution is obtained by means of numerical integration of appropriate boundary-value problems.

1. CONSTITUTIVE RELATIONS OF THE ELASTICITY THEORY AND FORMULATION OF BOUNDARY-VALUE PROBLEMS

1.1. Formulation of the Problem. Let us consider a cylindrical solid occupying the volume $V = S \times [0, L]$ (S is the cylinder cross section and L is its length). We denote the side surface by $\Gamma = \partial S \times [0, L]$, where ∂S is the boundary of the cross section S. We align the origin of the Cartesian coordinate system $Ox_1x_2x_3$ with the geometric center of gravity of one of the end faces of the cylinder and direct the Ox_3 axis along the cylinder centerline. This coordinate system will be called the basic coordinate system. To describe helical anisotropy, we introduce a helical cylindrical coordinate system (r, θ, z) related to the basic coordinate system by the expressions

$$x_1 = r\cos(\theta + \tau z), \qquad x_2 = r\sin(\theta + \tau z), \qquad x_3 = z, \tag{1.1}$$

where $\tau = \text{const}$.

The transition to the cylindrical coordinate system is made because the main attention will be paid below to solving the problem of a cylinder with a ring-shaped cross section $S = [r_1, r_2] \times [0, 2\pi]$ (r_1 and r_2 are the inner and outer radii of the cylinder, respectively).

At $r = \text{const}$ and $\theta = \text{const}$, relations (1.1) are parametric equations of the helical line, with $\tau = 2\pi/h$ (h is the helical pitch). The radius-vector of the points of the helical line is presented in the form

$$\boldsymbol{R} = r\boldsymbol{e}_1' + z\boldsymbol{e}_3',$$

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where

$$\boldsymbol{e}_1' = \boldsymbol{e}_r = \boldsymbol{i}_1 \cos(\theta + \tau z) + \boldsymbol{i}_2 \sin(\theta + \tau z),$$

$$\boldsymbol{e}_2' = \boldsymbol{e}_\theta = -\boldsymbol{i}_1 \sin(\theta + \tau z) + \boldsymbol{i}_2 \cos(\theta + \tau z), \qquad \boldsymbol{e}_3' = \boldsymbol{e}_z,$$

$\boldsymbol{i}_n$ are the orths of the basic coordinate system.

We relate the helical line to the natural basis (Frenet reference frame)

$$\boldsymbol{e}_1 = \boldsymbol{n}, \qquad \boldsymbol{e}_2 = \boldsymbol{b}, \qquad \boldsymbol{e}_3 = \boldsymbol{t},$$

where $\boldsymbol{n}$, $\boldsymbol{b}$, and $\boldsymbol{t}$ are the orths of the principal normal, binormal, and tangential line, respectively. Using the formulas

$$\frac{d\boldsymbol{R}}{ds} = \boldsymbol{t}, \qquad \frac{d\boldsymbol{t}}{ds} = k\boldsymbol{n}, \qquad \boldsymbol{b} = \boldsymbol{t} \times \boldsymbol{n},$$

$$ds = g\,dz, \qquad g^2 = 1 + x^2, \qquad x = \tau r,$$

where $k = \tau^2 r/g^2$ is the curvature of the helical line, and applying some transformations, we obtain an orthogonal matrix of the transition from the basis $\boldsymbol{e}_j$ to the basis $\boldsymbol{e}_i'$:

$$A = \begin{Vmatrix} -1 & 0 & 0 \\ 0 & -1/g & x/g \\ 0 & x/g & 1/g \end{Vmatrix}.$$

We assume that the cylinder material in the basis $\boldsymbol{e}_i$ possesses rhombohedral symmetry. The relation between the stresses and strains is written in the vector–matrix form [7]:

$$\boldsymbol{\sigma} = C\boldsymbol{\varepsilon}, \qquad \boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_6)^{\mathrm{t}}, \qquad \boldsymbol{e} = (e_1, \ldots, e_6)^{\mathrm{t}}. \tag{1.2}$$

Here, we have

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ c_{12} & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ c_{14} & -c_{14} & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{44} & c_{14} \\ 0 & 0 & 0 & 0 & c_{14} & c_{66} \end{pmatrix},$$

$$\varepsilon_1 = \varepsilon_{11}, \quad \varepsilon_2 = \varepsilon_{22}, \quad \varepsilon_3 = \varepsilon_{33}, \quad \varepsilon_4 = 2\varepsilon_{34}, \quad \varepsilon_5 = 2\varepsilon_{13}, \quad \varepsilon_6 = 2\varepsilon_{12},$$

$$\sigma_1 = \sigma_{11}, \quad \sigma_2 = \sigma_{22}, \quad \sigma_3 = \sigma_{33}, \quad \sigma_4 = \sigma_{23}, \quad \sigma_5 = \sigma_{13}, \quad \sigma_6 = \sigma_{12},$$

ε_{ij} and σ_{ij} are the components of the tensors of small strains and stresses, respectively.

Let us denote the stress vector, strain vector, and matrix of elasticity moduli in the basis of the helical coordinate system $\boldsymbol{e}_j'$ by $\boldsymbol{\sigma}'$, $\boldsymbol{\varepsilon}'$, and C', respectively. Hooke's law acquires the form

$$\boldsymbol{\sigma}' = C'\boldsymbol{\varepsilon}', \qquad C' = (c_{ij}') \quad (i = 1, \ldots, 6,\ j = 1, \ldots, 6), \tag{1.3}$$

where

$$c_{11}' = c_{11}, \qquad c_{12}' = (c_{12} - 2c_{14}x + c_{13}x^2)/g^2,$$

$$c_{13}' = (c_{13} + 2c_{14}x + c_{12}x^2)/g^2, \qquad c_{14}' = [(c_{12} - c_{13})x - c_{14}(x^2 - 1)]/g^2,$$

$$c_{22}' = [c_{11} + 4c_{14}x + (c_{13} + 2c_{44})2x^2 + c_{33}x^4]/g^4,$$

$$c_{23}' = [c_{13} - 2c_{14}x + (c_{11} + c_{33} - 4c_{44})x^2 + 2c_{14}x^3 + c_{13}x^4]/g^4,$$

$$c_{24}' = [-c_{14} - (c_{13} + 2c_{44} - c_{11})x + 3c_{14}x^2 - (c_{33} - c_{13} - 2c_{44})x^3]/g^4,$$

$$c_{33}' = [c_{33} + 2(c_{13} + 2c_{44})x^2 - 4c_{14}x^3 + c_{11}x^4]/g^4,$$

$$c'_{34} = [-(c_{33} - c_{13} - 2c_{44})x - 3c_{14}x^2 - (2c_{44} + c_{13} - c_{11})x^3 + c_{14}x^4]/g^4,$$

$$c'_{44} = [c_{44} - 2c_{14}x + (-2c_{13} + c_{11} + c_{33} - 2c_{44})x^2 + 2c_{14}x^3 + c_{44}x^4]/g^4,$$

$$c'_{55} = (c_{44} + 2c_{14}x + c_{66}x^2)/g^2, \qquad c'_{56} = x[c_{14} - (c_{44} - c_{66})x - c_{14}x^2]/g^2,$$

$$c'_{66} = (c_{66} - 2c_{14}x + c_{44}x^2)/g^2.$$

The following relation between different notations is used below:

$$\sigma'_1 = \sigma_{rr}, \quad \sigma'_2 = \sigma_{\theta\theta}, \quad \sigma'_3 = \sigma_{zz}, \quad \sigma'_4 = \sigma_{\theta z}, \quad \sigma'_5 = \sigma_{rz}, \quad \sigma'_1 = \sigma_{r\theta},$$

$$\varepsilon'_1 = \varepsilon_{rr}, \quad \varepsilon'_2 = \varepsilon_{\theta\theta}, \quad \varepsilon'_3 = \varepsilon_{zz}, \quad \varepsilon'_4 = 2\varepsilon_{\theta z}, \quad \varepsilon'_5 = 2\varepsilon_{rz}, \quad \varepsilon'_6 = 2\varepsilon_{r\theta}.$$

The components of the strain tensor in the basis of the helical coordinate system are expressed via the coordinates of the displacement vector $\boldsymbol{u} = (u_r, u_\theta, u_z)^{\mathrm{t}}$ by the formulas

$$\begin{gathered} \varepsilon_{rr} = \partial_r u_r, \qquad \varepsilon_{\theta\theta} = (u_r + \partial_\theta u_\theta)/r, \qquad \varepsilon_{zz} = Du_z, \\ 2\varepsilon_{r\theta} = \partial_r u_\theta + (\partial_\theta u_r - u_\theta)/2, \qquad 2\varepsilon_{rz} = \partial_r u_z + Du_r, \\ 2\varepsilon_{z\theta} = \partial_\theta u_z + Du_\theta. \end{gathered} \tag{1.4}$$

In this case, the equilibrium equations in stresses have the form

$$\begin{gathered} \partial_r(r\sigma_{rr}) - \sigma_{\theta\theta} + \partial_\theta\sigma_{r\theta} + rD\sigma_{rz} = 0, \\ \partial_r(r\sigma_{r\theta}) + \sigma_{r\theta} + \partial_\theta\sigma_{\theta\theta} + rD\sigma_{\theta z} = 0, \\ \partial_r(r\sigma_{rz}) + \partial_\theta\sigma_{\theta z} + rD\sigma_{zz} = 0. \end{gathered} \tag{1.5}$$

In Eqs. (1.4) and (1.5), we have

$$\partial_r = \frac{\partial}{\partial r}, \qquad \partial_\theta = \frac{\partial}{\partial \theta}, \qquad \partial = \frac{\partial}{\partial z}, \qquad D = \partial - \tau\,\partial_\theta.$$

We assume that the side surface of the cylinder is free from stresses:

$$r = r_\alpha \quad (\alpha = 1, 2): \qquad \sigma_{rr} = 0, \quad \sigma_{r\theta} = 0, \quad \sigma_{rz} = 0. \tag{1.6}$$

1.2. Vector–Operator Form of the Problem. The problem posed can be presented in a vector–operator form as

$$M(\partial, \tau)\boldsymbol{u} \equiv \partial^2 A_0\boldsymbol{u} + \partial A_1\boldsymbol{u} + A_2\boldsymbol{u} = 0; \tag{1.7}$$

$$N(\partial, \tau)\boldsymbol{u} \equiv (\partial B_0\boldsymbol{u} + B_1\boldsymbol{u})\Big|_\Gamma = 0. \tag{1.8}$$

Equations (1.7) and (1.8) are the equilibrium equations and the boundary conditions, where A_k and B_i ($k = 0$, 1, and 2; $i = 0$ and 1) are the matrix differential operators over the variables r and θ of the zeroth, first, and second orders. The particular form of the operators A_k and B_i is not given here. We only note that, by virtue of Eqs. (1.3), the coefficients of these operators depend on r and τ, but do not depend on z, which allows us to seek for the solution in the form

$$\boldsymbol{u} = \boldsymbol{a}\,\mathrm{e}^{\gamma z}.$$

As a result, we obtain a spectral problem on the cross section $z = \mathrm{const}$

$$M_1(\gamma)\boldsymbol{a} \equiv \{M(\gamma)\boldsymbol{a}, N(\gamma)\boldsymbol{a}\} = 0. \tag{1.9}$$

According to the general theory of quadratic pencils of symmetric operators [8], the spectrum of the operator $M_1(\gamma)$ is discrete, has an accumulation point at infinity, and is located in the complex plane $\gamma = \alpha + i\beta$ symmetrically with respect to the real axis, i.e., at $\alpha \neq 0$, any eigenvalue $\gamma_s^+ = \gamma_s = \alpha_s + i\beta_s$ ($\alpha_s \geq 0$ and $\beta_s \geq 0$) corresponds to three eigenvalues $\gamma_{-s}^+ = \alpha_s - i\beta_s$, $\gamma_s^- = -\gamma_s$, and $\gamma_{-s}^- = -\alpha_s - i\beta_s$. It was shown [1, 3, 4] that $\gamma_0 = 0$ and $\gamma_1^\pm = \pm i\tau$ are quadruple eigenvalues, and there are no purely imaginary eigenvalues except for $\gamma_1^\pm$.

The solution of problem (1.9) can be presented in the form [4]

$$\boldsymbol{u} = \boldsymbol{u}_{\mathrm{S}} + \boldsymbol{u}_p,$$

where $\boldsymbol{u}_{\mathrm{S}}$ is the Saint-Venant solution corresponding to the eigenvalues $\gamma_0 = 0$ and $\gamma_1^{\pm} = \pm i\tau$ and $\boldsymbol{u}_p$ is the solution corresponding to the remaining part of the spectrum and having the form

$$\boldsymbol{u}_p = \sum_k [C_k^- u_k^-(z) + C_k^+ u_k^+(z-L)], \qquad u_k^{\pm}(z) = a_k^{\pm} \exp{(\gamma_k^{\pm} z)},$$

where $C_k^{\pm}$ are arbitrary constants.

The solution $\boldsymbol{u}_S$ in the "principal" solution, because it covers the entire area occupied by the cylinder; the solution $\boldsymbol{u}_p$ is the "boundary layer," because it is localized near the end faces of the cylinder $z = 0, L$ and decreases exponentially with distance from the end faces.

It should be noted that the stress state of the Saint-Venant solution in an arbitrary cross section $z = \mathrm{const}$ in the integral meaning is equivalent to the principal vector and the principal moment of external forces applied to one end face of the cylinder. The principal vector and the principal moment of stresses corresponding to all values of $u_k^{\pm}(z)$ are equal to zero.

2. ELEMENTARY SAINT-VENANT SOLUTIONS OF THE TENSION–TORSION PROBLEM

The Saint-Venant solution of the tension–torsion problem [3, 4] is a linear combination of elementary solutions corresponding to the quadruple eigenvalue $\gamma_0 = 0$ and can be presented as

$$\boldsymbol{u}_{\mathrm{S}} = \sum_{l=1}^{4} X_l \boldsymbol{u}_l,$$

$$\boldsymbol{u}_1 = (0,0,1)^{\mathrm{t}}, \qquad \boldsymbol{u}_2 = (0,r,0)^{\mathrm{t}}, \qquad \boldsymbol{u}_3 = z\boldsymbol{u}_1 + \boldsymbol{a}_3, \qquad \boldsymbol{u}_4 = z\boldsymbol{u}_2 + \boldsymbol{a}_4,$$

$$\boldsymbol{a}_s = (a_{r,s}, a_{\theta,s}, a_{z,s})^{\mathrm{t}}, \qquad s = 3,4.$$

Here, $\boldsymbol{a}_s$ are the vector-functions whose coordinates depend on r and are determined by solving the boundary-value problems given below, X_1 is a constant that has the meaning of displacement of the cylinder as a solid along the Oz axis, X_2 is a constant that has the meaning of the angle of rotation about the Oz axis, and X_3 and X_4 are constants whose mechanical meaning is explained below.

Let us consider the problem of determining the vector-functions $\boldsymbol{a}_s$. Taking into account Eqs. (1.3), we determine the stresses corresponding to the vectors $\boldsymbol{u}_s$:

$$\boldsymbol{\sigma}'_s = C'\boldsymbol{\varepsilon}_s^{(1)} + C'\boldsymbol{\varepsilon}_s^{(0)},$$

$$\boldsymbol{\varepsilon}_s^{(1)} = \Big(\frac{da_{r,s}}{dr}, \frac{a_{r,s}}{r}, 0, \frac{da_{z,s}}{dr}, \frac{da_{\theta,s}}{dr} - \frac{a_{\theta,s}}{r}\Big)^{\mathrm{t}}, \tag{2.1}$$

$$\boldsymbol{\varepsilon}_3^{(0)} = (0,0,1,0,0,0)^{\mathrm{t}}, \qquad \boldsymbol{\varepsilon}_4^{(0)} = (0,0,0,r,0,0)^{\mathrm{t}}.$$

As the coordinates of the vectors $\boldsymbol{\sigma}'_s$ depend only on r, we use the equilibrium equations (1.5) and the boundary conditions (1.6) to obtain

$$\partial_r(r\sigma_{rr,s}) - \sigma_{\theta\theta,s} = 0, \qquad \sigma_{rr,s}(r_\alpha) = 0,$$

$$\partial_r(r\sigma_{r\theta,s}) + \sigma_{r\theta,s} = 0, \qquad \sigma_{r\theta,s}(r_\alpha) = 0, \tag{2.2}$$

$$\partial_r(r\sigma_{rz,s}) = 0, \qquad \sigma_{rz,s}(r_\alpha) = 0,$$

whence it follows that

$$\sigma_{r\theta,s} = \sigma_{rz,s} = 0.$$

From these relations and Eqs. (2.1) for $\sigma_{r\theta,s}, \sigma_{rz,s}$, we obtain

$$a_{\theta,s} = X_{1,s} r + X_{0,s}, \qquad a_{z,s} = X_{2,s}$$

($X_{0,s}$, $X_{1,s}$, and $X_{2,s}$ are arbitrary constants, which can be set equal to zero).

Substituting the relations for $\sigma_{rr,s}$ and $\sigma_{\theta\theta,s}$ from Eqs. (2.1) into Eqs. (2.2), we obtain the boundary-value problems for determining $a_{r,s}$

$$Za_{r,s} = F_s, \qquad la_{r,s}\Big|_{r=r_\alpha} = f_{\alpha,s}, \tag{2.3}$$

where

$$Za_s = \frac{d}{dr}\Big(rc'_{11}\frac{da_s}{dr} + c'_{12}a_s\Big) - c'_{12}\frac{da_s}{dr} - \frac{1}{r}c'_{22}a_s, \qquad la_s = c'_{11}\frac{da_s}{dr} + \frac{1}{r}c'_{12}a_s,$$

$$s = 3: \qquad F_3 = -\frac{d(rc'_{13})}{dr} + c'_{23}, \qquad f_{\alpha,3} = -c'_{13}(r_\alpha),$$

$$s = 4: \qquad F_4 = -\frac{d(r^2c'_{14})}{dr} + rc'_{24}, \qquad f_{\alpha,4} = -r_\alpha c'_{14}(r_\alpha).$$

Let us determine the constants X_l ($l = 1, \ldots, 4$). We assume that the end faces of the cylinder are subjected to the boundary conditions

$$z = 0: \qquad u_r = u_\theta = u_z = 0; \tag{2.4}$$

$$z = L: \qquad \sigma_{rz} = p_r, \qquad \sigma_{z\theta} = p_\theta, \qquad \sigma_{zz} = p_z, \tag{2.5}$$

where the functions p_r, p_θ, and p_z depend only on r.

We assume that the specified external stresses are equivalent to the tensile force P_z and the torsion moment M_z:

$$2\pi\int_{r_1}^{r_2} p_r r\,dr = 0, \quad 2\pi\int_{r_1}^{r_2} p_\theta r\,dr = 0, \quad 2\pi\int_{r_1}^{r_2} p_z r\,dr = P_z, \quad 2\pi\int_{r_1}^{r_2} p_\theta r^2\,dr = M_z.$$

Obviously, we have

$$\sigma_{z\theta} = X_3\sigma_{z\theta,3} + X_4\sigma_{z\theta,4}, \qquad \sigma_{zz} = X_3\sigma_{zz,3} + X_4\sigma_{zz,4}$$

[$\sigma_{\theta z,s}, \sigma_{zz,s}$ are determined from Eqs. (2.1)]. Using these expressions to satisfy the boundary conditions (2.5) in the integral meaning, we obtain

$$d_{11}X_3 + d_{12}X_4 = P_z, \qquad d_{21}X_3 + d_{22}X_4 = M_z,$$

where

$$d_{11} = 2\pi\int_{r_1}^{r_2} r\sigma_{zz,3}\,dr, \qquad d_{22} = 2\pi\int_{r_1}^{r_2} r^2\sigma_{z\theta,4}\,dr,$$

$$d_{12} = 2\pi\int_{r_1}^{r_2} r\sigma_{zz,4}\,dr = d_{21} = 2\pi\int_{r_1}^{r_2} r^2\sigma_{z\theta,3}\,dr.$$

According to [4], we can assume that $X_1 = X_2 = 0$ under the boundary conditions (2.4) and $r_2/L \ll 1$.

3. METHODS OF CONSTRUCTING ELEMENTARY SAINT-VENANT SOLUTIONS AND RESULTS OF THE NUMERICAL ANALYSIS

3.1. Method of the Small Parameter. First, we construct analytical solutions. We transform Eqs. (1.2) and (1.3) by using the replacements $r = r_2\xi$ and $\tau_0 = r_2\tau$. Assuming that the dimensionless parameter is $\tau_0 \ll 1$, we expand c'_{ml} into series with respect to τ_0. As a result, we obtain the following expressions for the principal terms of these expansions:

$$c'_{11} = c_{11}, \qquad c'_{12} = c_{12} - 2c_{14}\tau_0\xi, \qquad c'_{13} = c_{13} + 2c_{14}\tau_0\xi,$$

$$c'_{14} = c_{14} + \tau_0\xi(c_{12} - c_{13}), \qquad c'_{22} = c_{11} + 4c_{14}\tau_0\xi,$$

$$c'_{23} = c_{13} - 2c_{14}\tau_0\xi, \qquad c'_{24} = -c_{14} - \tau_0\xi(c_{13} + 2c_{44} - c_{11}), \tag{3.1}$$

$$c'_{33} = c_{33}, \qquad c'_{34} = (-c_{33} + c_{13} + 2c_{44})\tau_0\xi, \qquad c'_{44} = c_{44} - 2c_{14}\tau_0\xi,$$

$$c'_{55} = c_{44} + 2c_{14}\tau_0\xi, \qquad c'_{56} = c_{14}\tau_0\xi, \qquad c'_{66} = c_{66} - 2c_{14}\tau_0\xi.$$

The solution of the boundary-value problems (2.3) is sought in the form

$$a_{r,s} = a_s^{(0)} + \tau_0 a_s^{(1)} + \ldots\ . \tag{3.2}$$

Substituting Eqs. (3.1) and (3.2) into Eqs. (2.3), applying some standard transformations, and integrating, we obtain the problems

$$a_{r,3} = -\nu'\xi + O(\tau_0^2),$$

$$a_{r,4} = -K_0r^2 + K_1R_2K_0\frac{1}{r} + K_0R_1r + O(\tau_0^2),$$

where

$$\nu' = \frac{c_{13}}{c_{11} + c_{12}}, \quad K_0 = \frac{c_{14}}{c_{11}}, \quad K_1 = \frac{c_{11} + c_{12}}{c_{11} - c_{12}}, \quad R_1 = \frac{r_1^2 + r_2^2 + r_1r_2}{r_1 + r_2}, \quad R_2 = \frac{(r_1r_2)^2}{r_1 + r_2}.$$

Let us give the expressions for the principal terms of the stress tensor components:

$$\sigma_{zz,3} = E' + O(\tau_0), \qquad \sigma_{\theta z,3} = O(\tau_0),$$

$$\sigma_{r\theta,l} = \sigma_{rz,l} = 0, \qquad \sigma_{rr,3} = \sigma_{\theta\theta,3} \equiv O(\tau_0^2),$$

$$\sigma_{rr,4} = K_0(c_{11} + c_{12})(-r - R_2/r^2 + R_1) + O(\tau_0),$$

$$\sigma_{\theta\theta,4} = K_0(c_{11} + c_{12})(-2r + R_2/r^2 + R_1) + O(\tau_0),$$

$$\sigma_{zz,4} = K_0c_{13}(-3r + 2R_1) + O(\tau_0),$$

$$\sigma_{\theta z,4} = r(c_{44} - K_0c_{14}) - 2K_0K_1c_{14}R_2/r^2 + O(\tau_0).$$

Here $E' = c_{33} - 2\nu'c_{13}$.

Remark 1. The principal terms of the formulas given above correspond to the solution of the problems of tension and torsion of a cylinder with cylindrical rhombohedral anisotropy. These problems were solved in [9]. In using the semi-inverse method, however, Gorodtsov and Lisovenko [9] made some essential mistakes and, consequently, obtained incorrect results. An attempt to correct these mistakes was made in [10].

3.2. Method of Numerical Integration of Boundary-Value Problems. To perform numerical integration, we present problem (2.3) in the form of a system of the first-order differential equations by introducing the vector $\boldsymbol{y}_s = (y_{1,s}, y_{2,s})^{\mathrm{t}}$ with the coordinates

$$y_{1,s} = a_{r,s}, \qquad y_{2,s} = r\sigma_{rr,s}/c_{11}.$$

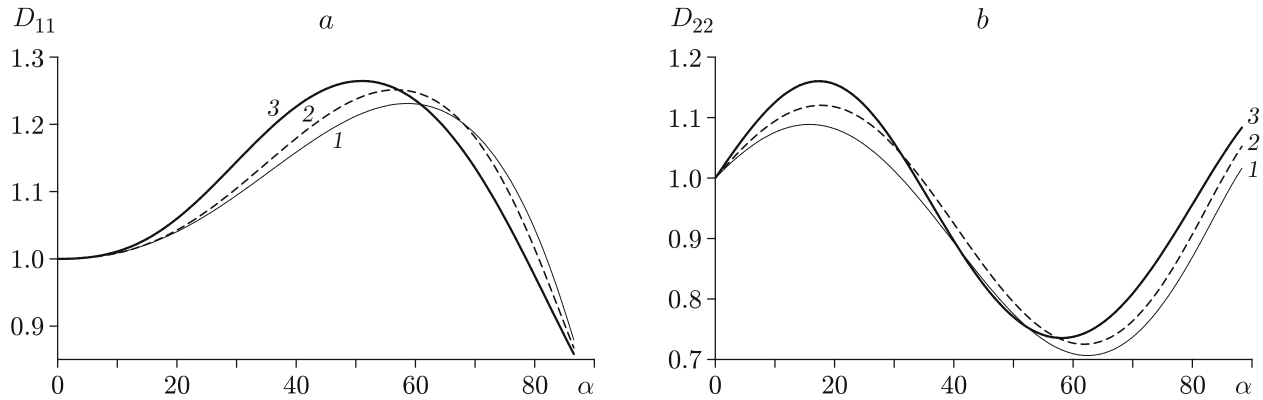

Fig. 1. Tensile stiffness D_{11} (a) and torsion stiffness D_{22} (b) versus the parameter α for $a = 0.1$ (1), 0.4 (2), and 0.8 (3).

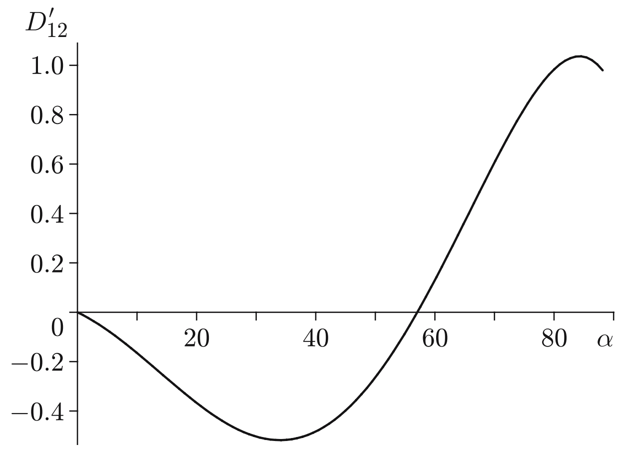

Fig. 2. Dependence $D'_{12}(\alpha)$ at $a = 0.4$.

Then, the boundary-value problems (2.3) can be written as

$$\frac{d\boldsymbol{y}_s}{dr} - A\boldsymbol{y}_s = \boldsymbol{q}_s, \qquad y_{2,s}(r_\alpha) = 0, \tag{3.3}$$

where the elements of the matrix A and the coordinates of the vectors $\boldsymbol{q}_s$ have the form

$$A_{11} = -\frac{c'_{12}}{rc'_{11}}, \qquad A_{12} = \frac{1}{rc'_{11}}, \qquad A_{21} = \frac{c'_{11}c'_{22} - c'^2_{12}}{rc'_{11}}, \qquad A_{22} = \frac{c'_{12}}{rc'_{11}},$$

$$q_{1,3} = -\frac{c'_{13}}{c'_{11}}, \qquad q_{2,3} = c'_{23} - \frac{c'_{12}c'_{13}}{c'_{11}}, \qquad q_{1,4} = -\frac{rc'_{14}}{c'_{11}}, \qquad q_{2,4} = r\Big(c'_{24} - \frac{c'_{12}c'_{14}}{c'_{11}}\Big).$$

Numerical integration of the boundary-value problems (3.3) was performed by the sweep method. The solution of problems (3.3) was sought in the form

$$\boldsymbol{y}_s = \boldsymbol{y}_s^0 + B_s\boldsymbol{y}_s^1,$$

where $\boldsymbol{y}_s^0$ and $\boldsymbol{y}_s^1$ are the solutions of the Cauchy problems

$$\frac{d\boldsymbol{y}_s^0}{dr} - A\boldsymbol{y}_s^0 = \boldsymbol{y}_s, \qquad \boldsymbol{y}_s^0(r_1) = (0, 0)^{\rm t},$$

$$\frac{d\boldsymbol{y}_s^1}{dr} - A\boldsymbol{y}_s^1 = 0, \qquad \boldsymbol{y}_s^1(r_1) = (1, 0)^{\rm t},$$

and the constants B_s are determined from the conditions

$$y_{2,s}(r_2) = B_s y^1_{2,s}(r_2) + y^0_{2,s}(r_2) = 0$$

for the solutions to satisfy the boundary conditions at $r = r_2$.

To conclude, we give some results of the numerical analysis of the problem. All calculations were performed for a cylinder with rhombohedral anisotropy in the Frenet reference frame with the following values of the elasticity moduli [11]: $c_{11} = 86.8 \cdot 10^9$ Pa, $c_{33} = 105.75 \cdot 10^9$ Pa, $c_{44} = 58.2 \cdot 10^9$ Pa, $c_{12} = 7.04 \cdot 10^9$ Pa, $c_{13} = 11.91 \cdot 10^9$ Pa, and $c_{14} = -18.04 \cdot 10^9$ Pa.

Based on numerical integration, we studied the dependences of the normalized elements of the stiffness matrix

$$D_{11} = d_{11}/d^0_{11}, \qquad D_{22} = d_{22}/d^0_{22}, \qquad D'_{12} = d_{12}/(r_2 d^0_{11}) \tag{3.4}$$

on the parameter $\alpha = \arctan \tau_0 \in [0, \pi/2]$ for different values of the parameter $a = r_1/r_2$. In Eqs. (3.4),

$$d^0_{11} = \pi E'(r_2^2 - r_1^2), \qquad d^0_{22} = \frac{\pi}{2}(r_2^4 - r_1^4)\Big(c_{44} - \frac{c_{14}^2}{c_{11}}\Big) - 4\pi \frac{c_{14}^2}{c_{11}} \frac{c_{12} + c_{11}}{c_{11} - c_{12}} \frac{(r_1 r_2)^2 (r_2 - r_1)}{r_1 + r_2}$$

are the tensile and torsion stiffnesses, respectively, of the cylinder considered as a rod at $\alpha = 0$.

Figure 1 shows the tensile stiffness D_{11} and the torsion stiffness D_{22} as functions of the parameter α. The dependence $D'_{12}(\alpha)$ is plotted in Fig. 2. It follows from Figs. 1 and 2 that the greatest tensile stiffness is observed for the values of the parameter α in the interval $[45^\circ, 65^\circ]$; the greatest torsion stiffness is observed in the range $[10^\circ, 25^\circ]$. The torsion stiffness also has a minimum in the range of α $[50^\circ, 65^\circ]$.

It should be noted that, for any fixed value of a, there exists such a value $\alpha = \alpha_*$ ($\alpha_* \neq 0, \pi/2$) at which d_{12} changes its sign to the opposite one. At this value of α (as at $\alpha = 0, \pi/2$), tension–compression of the cylinder does not lead to its torsion; vice versa, torsion does not lead to longitudinal deformation.

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